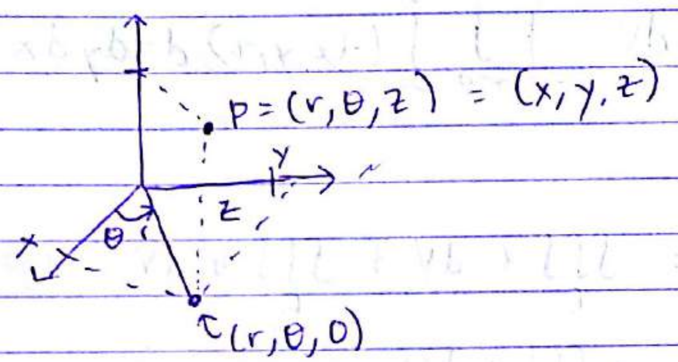
10/28/15

15.8 Triple Integrals in Cylindrical Coordinates

Cylindrical Coordinates:

If P is a point in 3D space, its cylindrical coordinates are (r, θ, z) where r, θ are the polar coordinates of the projection P on the xy plane, and z is the directed distance from the x-y plane to P:



$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$, $z = z$

Triple Integrals w/ Cylindrical Coordinates

Suppose E is a type 1 solid region in $\mathbb{R}^3$:

$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

D can easily be described in polar coordinates:

$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$

We know that

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

$$= \iint_D F(x,y)\,dA, \text{ where } F(x,y) = \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz$$

And that

$$\iint_D F(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

& $F(r\cos\theta, \sin\theta) = \displaystyle\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz$

Combining those formulas, we get:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, \underbrace{dz\, r\, dr\, d\theta}_{\text{also written as } r\,dz\,dr\,d\theta}}$$

example. Find the equation for the circular cone $\frac{z^2}{c^2} = x^2+y^2$ in cylindrical coordinates.

$z^2 = \underbrace{(x^2+y^2)}_{r^2}c^2$, so $\boxed{z = cr}$

example. Find the volume of the solid bound by the paraboloids $z = x^2+y^2$ and $z = 8-x^2-y^2$

$z = x^2+y^2 \Downarrow z = r^2$, $z = 8-x^2-y^2 \Downarrow z = 8-r^2$ } Meet when $r^2 = 8-r^2$

$r^2 = 4$, $\boxed{r=2,\ z=4}$

so we have: $E = \{(x,y,z) \mid (x,y) \in D,\ x^2+y^2 \le z \le 8-x^2-y^2\}$

$D = \{(x,y) \mid (x^2+y^2 \le 4)\}$

$= \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$

So, $V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}r^2\right]_{r^2}^{8-r^2} dr\,d\theta$ and so on...

example. Find the volume of the solid enclosed by $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 2 \Rightarrow z = \sqrt{2-r^2}$

$z = \sqrt{x^2+y^2} \Rightarrow z = r$, so $r \le z \le \sqrt{2-r^2}$, intersection when $r = 1$.

So, $E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r \le z \le \sqrt{2-r^2}\}$

& we get: $V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^1\int_1^{\sqrt{2-r^2}} r\,dr\,d\theta$ and so on...