10/26/15

15.7 Triple Integrals

- $f(x) \rightsquigarrow \int_a^b f(x)\,dx$ $[a,b] \subset \mathbb{R}$
- $f(x,y) \rightsquigarrow \iint_D f(x,y)\,dA$ $D \subset \mathbb{R}^2$
- $f(x,y,z) \rightsquigarrow \iiint_E f(x,y,z)\,dV$ $E \subset \mathbb{R}^3$

Triple integrals over rectangular box:

Let $f(x,y,z)$ be continuous on a rectangular box

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$
$$= [a,b] \times [c,d] \times [r,s]$$

- Partition $[a,b]$ into l subintervals of width Δx,
  $[c,d]$ into m subintervals of width Δy, and
  $[r,s]$ into n subintervals of width Δz

This gives us a partition of B into lmn sub-boxes (each of volume $\Delta V = \Delta x \Delta y \Delta z$)

- Choose a sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ in each sub-box B_{ijk}

- Form the Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

- Take the limit as $l,m,n \to \infty$ to get the triple integral.

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

Fubini's Theorem for triple integrals: (f is continuous on B)

$$\iiint_B f\,dV = \int_r^s\int_c^d\int_a^b f\,dx\,dy\,dz$$

$$= \int_r^s\int_a^b\int_c^d f\,dy\,dx\,dz = \int_c^d\int_r^s\int_a^b f\,dx\,dz\,dy = \int_c^d\int_a^b\int_r^s f\,dz\,dx\,dy$$

$$= \int_a^b\int_r^s\int_c^d f\,dy\,dz\,dx = \int_a^b\int_c^d\int_r^s f\,dz\,dy\,dx$$

Triple Integrals over General Solid Regions

Let $E \in \mathbb{R}^3$ be a bounded region. Enclose E in a box $B = [a,b] \times [c,d] \times [r,s]$.

Define $F: B \to \mathbb{R}$ by $F(x,y,z) \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \notin E \end{cases}$

Then we define

$$\boxed{\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV}$$

, much like we did for a double integral.

Solid regions of type 1, 2, 3:

Type 1

$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

→ - D is projection of E on xy-plane, E lies between u_1 & u_2.

Type 2

$E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$

→ -

Type 3

$E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$

→ -

(same definition)

Type 1:

$$\iiint_E f(x,y,z) = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

If D is of type I:

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

If D is Type II:

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Type 2

$$\iiint_E f(x,y,z) = \iint_D \left[\int_{u_2(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

Type 3

$$\iiint_E = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

can be converted like done in type 1.

Applications of Triple Integrals

- Volume: $V(E) = \iiint_E dV$ (similar to $\int_a^b dx = b-a$, $\iint_D dA = A(D)$)

- Density & Mass:

Mass of a solid with density $\rho(x,y,z)$:

$$m = \iiint_E \rho(x,y,z)\,dV$$

Moments / Cent. of Mass (C.O.M):

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV \qquad M_{xy} = \iiint_E z\rho(x,y,z)\,dV$$

$$M_{xz} = \iiint_E y\rho(x,y,z)\,dV$$

C.O.M is $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = \frac{M_{yz}}{m}$ $\bar{y} = \frac{M_{xz}}{m}$ $\bar{z} = \frac{M_{xy}}{m}$

(If ρ is constant, the C.O.M. is the centroid).

Moments of Inertia:

$$I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\,dV$$

dist from (x,y,z) to x-axis is $\sqrt{y^2+x^2}$.

$$I_y \iiint_E (x^2+z^2)\rho(x,y,z)\,dV$$

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV$$

Probability: If X, Y, Z are continuous random variables, their joint density function is $f(x,y,z)$ such that

- $P((X,Y,Z) \in E) = \iiint_E f(x,y,z)\,dV$
- $f(x,y,z) \geq 0$
- $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = 1$

Properties

- $\iiint_E [f+g]\,dV = \iiint_E f\,dV + \iiint_E g\,dV$
- $\iiint_E cf\,dV = c\iiint_E f\,dV$
- If $f \geq g$ on E, then $\iiint_E f\,dV \geq \iiint_E g\,dV$
- If $m \leq f \leq M$ on E, then $mV(E) \leq \iiint_E f\,dV \leq MV(E)$
- If $E = E_1 \cup E_2$ and E_1, E_2 only overlap on their boundaries, $\iiint_E f\,dV = \iiint_{E_1} f\,dV + \iiint_{E_2} f\,dV$.