10/23/15

15.6 Surface Area

we can find SA of a surface with equation $z=f(x,y)$.

Flashback: Area of a surface of revolution.

- If the surface S is obtained by rotating the graph of $y=f(x)\geq 0$, $a\leq x\leq b$ about the x-axis, then its area is:

$$A(S) = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

$$= \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_a^b 2\pi y\,ds, \text{ where } ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

(assuming $f'(x)$ is continuous.)

- Similarly if S is obtained by rotating the graph of $x=g(y)$, $c\leq y\leq d$ still about the x-axis, we have

$$A(S) = \int_c^d 2\pi y\sqrt{1+g'(y)^2}\,dy$$

$$= \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_c^d 2\pi y\,ds, \text{ where } ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

- If we rotate about the y-axis we get:

$A(S)=\int 2\pi x\,ds$ where $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or

$\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

- $A(S)$ where S is $z=f(x,y)$ over a region D:

Let $f(x,y) \geq 0$ on D. Assume f_x & f_y are continuous.

Let S be the graph of f over D:

$$S = \{(x,y,z) \mid (x,y) \in D,\ z = f(x,y)\}$$

$A(S)$ is the area of S.

- We can assume $D = R = [a,b] \times [c,d]$
- Partition D into small subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$ with area $\Delta A = \Delta x \Delta y$.
- Let $(x_i^*, y_j^*) = (x_{i-1}, y_{j-1})$ and let $P_{ij} = (x_i^*, y_j^*, f(x_i^*, y_j^*)) \in S$
- The tangent plane to S at P_{ij} is a good approx. to S near P_{ij}. Let ΔT_{ij} = are of parallelogram on this tangent plane that lies above R_{ij}.

- $\Delta T_{ij} \approx \Delta S_{ij}$ = area of part of S that lies over R_{ij}.

So $\sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$ approximates the total area of S.

So the "formal", unuseful for $A(S)$ is:

$$\boxed{A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}}$$

The Helpful Version

Let $\vec{a}, \vec{b}$ be the vectors starting at P_{ij} and lying on the sides of the above parallelogram w/ the area ΔT_{ij}

Area $= |\vec{a} \times \vec{b}|$

* $f_x(x_i^*, y_j^*)$ and $f_y(x_i^*, y_j^*)$ are slopes to tangent lines in P_{ij} in dir. ∥ to x & y-axes.

Therefore: $\vec{a} = \Delta x \hat{\imath} + f_x(x_i^*, y_j^*)\Delta x \ \hat{k}$

$\vec{b} = \Delta y \hat{\jmath} + f_y(x_i^*, y_j^*)\Delta y \ \hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Delta x & 0 & \Delta x f_x(x_i^*, y_j^*) \\ 0 & \Delta y & \Delta y f_y(x_i^*, y_j^*) \end{vmatrix}$$

$$= -f_x(x_i^*, y_j^*)\Delta x \Delta y \hat{\imath} - f_y(x_i^*, y_j^*)\Delta x \Delta y \hat{\jmath} + \Delta x \Delta y \hat{k}$$

$$= (-f_x(x_i^*, y_j^*)\hat{\imath} - f_y(x_i^*, y_j^*)\hat{\jmath} + \hat{k})\Delta A$$

$$\therefore \Delta T_{ij} = |\vec{a} \times \vec{b}| = \sqrt{(f_x(x_i^*, y_j^*))^2 + (f_y(x_i^*, y_j^*))^2 + 1} \ \Delta A$$

And $A(S) = \iint_D \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1} \ dA$

$$\boxed{= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \ dA}$$

similar to $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx$