10/14/15

15.2 Iterated Integrals

The double integrals $\iint_R f(x,y)\,dA$ over rectangles (or over more complex regions) are very difficult to evaluate from their definition.

Fortunately, we can often express a double integral as an iterated integral, which allows us to calculate it by evaluating two single integrals.

Partial Integration

Let $f(x,y)$ be integrable over the rectangle
$R = [a,b] \times [c,d]$
- If we fix x, we can integrate $f(x,y)$ w/ respect to y from c to d: $\int_c^d f(x,y)\,dy$

This is called partial integration with respect to y.

If we now vary x, we get a function $A(x) = \int_c^d f(x,y)\,dy$ because the integral depends on x.

We can integrate $A(x)$ from a to b, w/ respect to x

$$\boxed{\int_a^b A(x)\,dx = \underbrace{\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx}_{\text{this is called an iterated integral}}}$$

Similarly, we can fix y, we can integrate $f(x,y)$ w/ respect to x from a to b: $\int_a^b f(x,y)\,dx$
This integral depends on y, so we get a function:
$B(y) = \int_a^b f(x,y)\,dx$. Integrate $B(y)$ from c to d to get:

$$\int_c^d B(y)\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Notation: the brackets are usually omitted, so we get:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \quad \text{and} \quad \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- we work from the inside-out to evaluate these.

Fubini's Theorem: Suppose f is continuous on $R=[a,b]\times[c,d]$

Then, $$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- This theorem is true under the weaker hypothesis that f is bounded on R and continuous except maybe on a finite # of smooth curves, and that the iterated integrals exist.

- Think of this as a counterpart to Clairaut's Theorem

Proof - when $f \geq 0$. Suppose $f(x,y) \geq 0 \ \forall x \in R = [a,b]\times[c,d]$ and f is continuous on R. Let J = solid that lies above R and under the surface $z=f(x,y)$. This surface is the graph of f over R.

We saw in 15.1 that $\iint f(x,y)\,dA = V =$ Volume of J.

From Math 1A we also know that $V = \int_a^b A(x)\,dx$, where $A(x)$ is the area of the cross-section of J in the plane through x and $\perp$ to x-axis.

→ So $A(x_0)$ = area of crossection of J in the plane $x=x_0$

The curve C has the eqn $z=f(x,y)$, where x is held constant and $c \leq y \leq d$. So, $A(x) = \int_c^d f(x,y)\,dy$

Therefore, $$\iint_R f(x,y)\,dA = V = \int_a^b A(x)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

In the same manner, but using cross-sections $\perp$ to the y-axis, we get

$$\iint_R f(x,y)\,dA = V = \int_c^d \underset{\downarrow}{B(y)}\,dy = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b f(x,y)\,dx$$

Remark

Fubini's theorem tells us that we get the same result regardless of the order in which we perform the partial integration. However, from the practical point of view, one order could result in an easier calculation than the other.

Special Case

Suppose $f(x,y) = g(x)h(y)$. Then $\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy$

$$= \int_c^d \left[\int_a^b g(x)h(y)\,dx\right]dy$$

$$= \int_c^d \left[h(y)\int_a^b g(x)\,dx\right]dy \leftarrow h(y) \text{ is constant in } \int_a^b g(x)\,dx$$

$$= \int_c^d h(y)\,dy \cdot \int_a^b g(x)\,dx \leftarrow \int_a^b g(x) \text{ is constant.}$$