10/12/15

15.1 Double Integrals Over Rectangles

Flashback: $\int_a^b f(x)\,dx$

Let $f(x)$ be continuous on $[a,b]$

- Partition $[a,b]$ into n subintervals:

$$a = x_0 < x_1 < x_2 < \ldots x_{i-1} < x_i < \ldots < x_{n-1} < x_n = b$$

The sub-intervals are $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$

Let $\Delta x_i = x_i - x_{i-1}$ (legth of subinterval)

In fact, we can restrict our attention to regular partitions (subintervals of equal length), $\Delta x = \frac{1}{n}(b-a)$

- Pick sample points: choose any point $x_i^* \in [x_{i-1}, x_i]$ $\forall\, i = 1, \ldots, n$
- Form the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$
- Take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \;\underline{\text{def}}\; \int_a^b f(x)\,dx$$

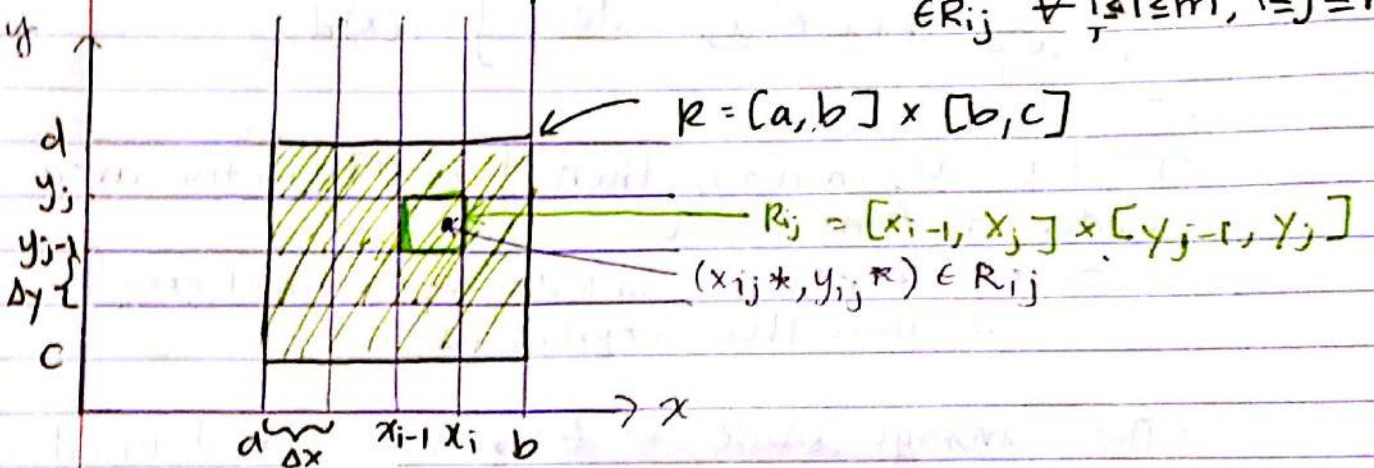
- If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ is the area under f from a to b.
  - → The Riemann sums are approximations to this area that improve as $n \to \infty$.
- The average value of f on $[a,b]$ is defined as:

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

let $f(x,y)$ be continuous on a rectangle
$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$

(bounded on R & continuous)

- Partition R into subrectangles:
  - Divide $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal length $\Delta x = \frac{1}{m}(b-a)$
  - Divide $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of equal length $\Delta y = \frac{1}{n}(d-c)$
  - Draw the lines $x = x_i$ and $y = y_j$. You get subrectangles.
- $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $1 \le i \le m$, $1 \le j \le n$
  - each R_{ij} has area $\Delta A = \Delta x \Delta y$
- Pick sample points: choose points $(x_{ij}^*, y_{ij}^*) \in R_{ij}$ $\forall 1 \le i \le m, 1 \le j \le n$



- Form Riemann Sum:

$$\sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} f(x_{ij}^*, y_{ij}^*)\Delta A = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

- Take the Limit as $m \to \infty$ and $n \to \infty$:

$$\underbrace{\boxed{\iint_R f(x,y)\,dA}}_{\text{double integral of } f \text{ over } R.} = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

NOTE: It does not matter what sample pts we use. So we could write $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$

upper right corner of R_{ij}

- Choosing (x_{ij}^*, y_{ij}^*) = center point of $R_{ij} = (\bar{x}_i, \bar{y}_j)$, the Riemann sums are the Midpoint Rule Approx.:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A$$

where $\bar{x}$ = midpt (x_{i-1}, x_i) & $\bar{y}$ = midpt (y_{j-1}, y_j)

- If $f(x,y) \geq 0$ on R, then the graph of f is a surface w/ eqn $z = f(x,y)$. Let S be the solid that lies above R and under the graph of f:

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in R \ \& \ 0 \leq z \leq f(x,y)\}$$

- The volume of S is defined to be $V = \iint_R f(x,y)\,dA$.
- The avg. value of f on R is:

$$\boxed{f_{avg} = \frac{1}{A(R)} \iint_R f(x,y)\,dA}$$ where $A(R)$ = area of R.

Since $A(R) f_{avg} = \iint_R f(x,y)\,dA$, the box w/ base R & height f_{avg} has the same volume as $\iint_R f(x,y)\,dA$. (I)

Properties of Double Integrals

① $\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

② $\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$, c is a constant.

③ If $f(x,y) \geq g(x,y)\ \forall\ (x,y) \in R$, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

~~a~~ assuming f, g are integrable.

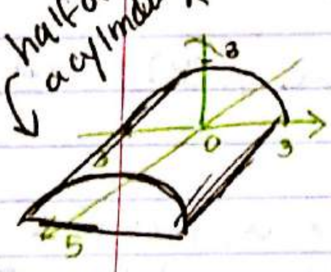
examples

Find the avg. val of the function $f(x,y) = \sqrt{9-y^2}$ on the rectangle $R = [0,5] \times [-3,3]$

$A(R) = 5 \times 6 = 30$

$\iint_R f(x,y)\,dA = \text{Vol. of } J = \frac{1}{2}\pi(3)^2(5) = \frac{45\pi}{2}$

half of a cylinder



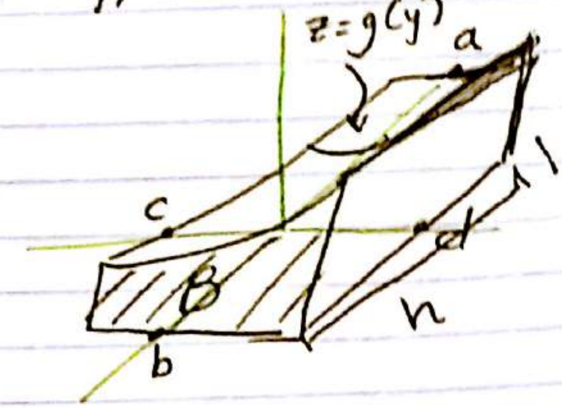
$\therefore f_{avg} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{45\pi}{30\cdot 2} = \boxed{\frac{3\pi}{4}}$

Let's generalize the prev. example to find $\iint_R f(x,y)\,dA$ when $f(x,y) = g(y) > 0$, $R = [a,b] \times [c,d]$

Here $\iint_R f(x,y)\,dA = V =$ volume of T, which is a "prism"

$V = Bh$, where B is the area of its base and h is its height. So

$$\iint_R f(x,y)\,dA = Bh = \left(\int_c^d g(y)\,dy\right)(b-a)$$



Similarily, if $f(x,y) = g(x) \geq 0$ on R, we have

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)(d-c)$$