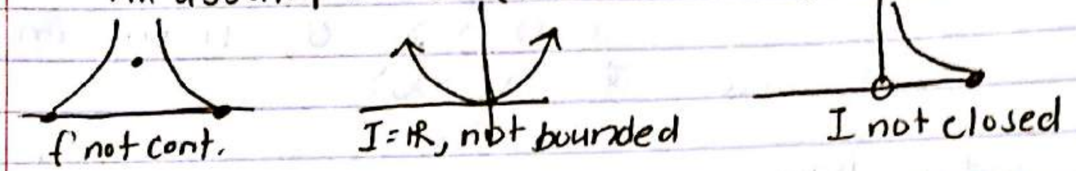
14.7 Maximum & Minimum Values

Flashback:

- The extreme value theorem for a function of one variable says that if the function is continuous on a closed, bounded interval I, thee is a max/min value on I.
  - All assumptions (f is continuous, I is bounded & closed)



To find the (absolute) max & min values of f on I:
- find $f(c)$ for all critical pts ($f'(c)=0$ or DNE) on I
- find $f(a)$, $f(b)$ at the endpoints of $I=[a,b]$
- The largest is (absolute) max. Smallest is (absolute) min.

Extreme values for functions of two variables

- f has a local maximum at (a,b) if for some disk centered at (a,b) — call it D — we have $f(x,y) \le f(a,b)\ \forall (x,y) \in D$
- $f(a,b)$ is a local max value. Similarly, for a disk D centered at (a,b), we have $f(x,y) \ge f(a,b)\ \forall (x,y) \in D$, then f has a local min at (a,b), and $f(a,b)$ is a local min. value.

- Let $S \subset$ dom f. f has an absolute maximum on S at (a,b) if $f(a,b) \ge f(x,y)\ \forall (x,y) \in S$ and an absolute minimum on S at (a,b) if $f(x,y) \ge f(a,b)\ \forall (x,y) \in S$.

Theorem: If f has a local max or min at (a,b) and if f_x, f_y exist @ (a,b), then $f_x(a,b) = 0 = f_y(a,b)$

Proof of Theorem: let $g(x) = f(x,b)$ and $G(y) = f(a,y)$
By Fermat's Theorem, $g'(a) = 0$, $G'(b) = 0$. But, $g'(a) = f_x(a,b)$ and $G'(b) = f_y(a,b)$. woo.

(a,b) is a critical point of f if $f_x(a,b) = 0 = f_y(a,b)$ or if one (or both) of the partial derivatives DNE.
* a critical pt is not necessarily a local max/min. but all local max/min are critical points.

Second derivatives test:
Suppose $f_x(a,b) = 0 = f_y(a,b)$ and f_{xx}, f_{yy}, and f_{xy} are continuous on some disk centered at (a,b). Let $D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2$.
- if $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min
- if $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.
- If $D < 0$, then $f(a,b)$ is neither a local max/min.
  - it's a "saddle point".

Note: If $D = 0$, the test gives no info.
D is a determinant: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

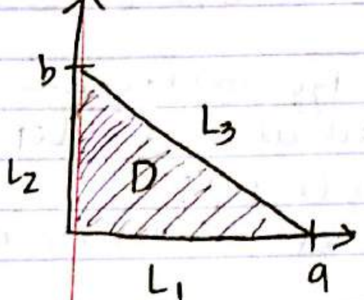
Extreme Value Theorem for Functions of Two Variables
If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains an absolute max/min on D. "D is closed" means it contains all of its boundary points.

* f must be continuous, D must be closed and bounded.

To find absolute max/min,
- find $f(a,b)$ for all crit. pts $(a,b) \in D$
- find extreme values of f on "∂D", the boundary of D.
- The largest value is the abs. max, smallest is abs. min.

Remark: Finding the extreme values of f on the boundary of D often reduces to a single variable problem, because the equation(s) of the boundary allow you to express x & y in terms of each other.

ex: D is the triangular region with vertices (0,0), (a,0), (0,b), say a, b > 0.



The boundary is the union of the line segments L_1, L_2, L_3.

L_1: $y=0$, $f(x,y) = f(x,0)$, $0 \le x \le a$

L_2: $x=0$, $f(x,y) = f(0,y)$, $0 \le y \le b$

L_3: $bx+ay=ab$ or $y = -b/a\, x + b$,
$f(x,y) = f(x, -\frac{b}{a}x+b)$, $0 \le x \le a$

Examples

#46 Find the dimensions of a box w/ V $1000\,cm^3$ that has min. SA.

x, y, z = dimensions. We must minimize $2xy+2xz+2yz$, or equivalently $xy+xz+yz$.

Now $V = 1000 = xyz$, so $z = \frac{1000}{xy}$. We will minimize

$f(x,y) = xy + \frac{1000}{y} + \frac{1000}{x}$. We have:

$f_x(x,y) = y - \frac{1000}{x^2}$ $\quad f_y = x - \frac{1000}{y^2}$. If $f_x(x,y) = 0$, then

$y = \frac{1000}{x^2}$. Substituting this into $f_y = x - \frac{1000}{(\frac{1000}{x^2})^2} = x - \frac{x^4}{1000}$

or (since $x \neq 0$), $x^3 = 1000$. $\therefore \boxed{x = 10}$, $\boxed{y} = \frac{1000}{10^2} = \boxed{10}$, and

$\boxed{z} = \frac{1000}{10 \cdot 10} = \boxed{10}$

#53 What is the largest possible volume of a box if length of its diagonal is a constant L?

Let x, y, z = dimensions. Then $L = \sqrt{x^2+y^2+z^2}$, so $z = \sqrt{L^2-x^2-y^2}$ and we need to maximize $V = xyz$
$= f(x,y) = xy\sqrt{L^2-x^2-y^2}$.

$$V_x = y\sqrt{L^2-x^2-y^2} - \frac{x^2y}{\sqrt{L^2-x^2-y^2}}$$

$x, y > 0$ and $x^2+y^2 < L^2$

$$V_y = x\sqrt{L^2-x^2-y^2} - \frac{xy^2}{\sqrt{L^2-x^2-y^2}}$$

$V_x = 0 \Rightarrow x^2\cancel{y} = \cancel{y}(L^2-x^2-y^2) \Rightarrow x^2 = L^2-x^2-y^2 \Rightarrow 2x^2 = L^2-y^2$

Similarly,
$V_y = 0 \Rightarrow 2y^2 = L^2-x^2$.

$$\begin{cases} 2x^2+y^2 = L^2 \\ 2y^2+x^2 = L^2 \end{cases} \Rightarrow 3x^2 = L^2 \Rightarrow x = \frac{L}{\sqrt{3}}, \text{ similarly } y = \frac{L}{\sqrt{3}}.$$

The only crit. point is $\left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right)$ and it must give a maximum. The max volume is: $\boxed{\frac{L^3}{3\sqrt{3}}}$