09/30/15

14.4 Tangent Planes and Linear Approximations

New policy: you can replace lowest midterm score w/ a higher final exam score

Flashback:

If f is a differential function of 1 variable, $f'(x_0)$ is the slope of the tangent line to the graph f at (x_0, y_0). (where $y_0 = f(x_0)$

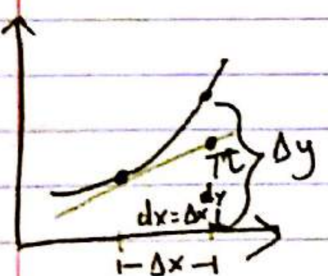
$\therefore$, an eqn for this line is: $y - y_0 = f'(x_0)(x - x_0)$

A linear approximation to $y = f(x)$ is: $y \approx f(x_0) + f'(x_0)(x - x_0)$

If x changes from a to $a + \Delta x$, the increment of $y = f(x)$ is $\Delta y = f(a + \Delta x) - f(a)$. If $f(x)$ is differentiable at a, we have $\Delta y = f'(a)\Delta x + \varepsilon \Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$.

because $\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon$ and $\frac{\Delta y}{\Delta x} \to f'(a)$ as $\Delta x \to 0$.

We define dx to be independent, and the differential of y is: $dy = f'(x)\,dx$



dy is approximation

Δy is actual Δy due to Δx.

Functions of Two variables: Tangent Planes

We use tangent planes!

let f be a funct. of 2 variables, with f_x, f_y, and is continuous.

S is graph of $z = f(x,y)$

Recall slicing surface w/plane in 14.3. to find the partial derivative.

By definition, the tangent plane at point P to be the plane that contains tangent lines T_1 and T_2.

We know from Ch. 12 the eqn of a plane is:

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

or

$$z - z_0 = a(x-x_0) + b(y-y_0) \quad a = -A/C, \; b = -B/C$$

if $y = y_0$,

$z - z_0 = a(x - x_0)$ This is the eqn of T_1.

So, we know $a = f_x(x_0, y_0)$

Similarily, setting $x = x_0$, we get $z - z_0 = b(y - y_0)$ and $b = f_y(x_0, y_0)$.

Therefore, if f_x and f_y are continuous, the eqn of a tangent plane to the surface $z = f(x,y)$ at $P = (x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linearization of f at (x_0, y_0) is the linear function:
$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ & the linear approximation is:
$f(x,y) \approx L(x,y)$

In general, such functions w/ good linear approximations are differentiable. Definition:
Let x change from a to $a + \Delta x$ & y change from b to $b + \Delta y$.
The corresponding increment of $z = f(x,y)$ is $\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$.
f is differentiable at (a,b) if: $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$
where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem: If f_x, f_y exist near (a,b) and are continuous at $f(a,b)$, then f is differentiable at (a,b).

<u>Differentials</u>

Let $z=f(x,y)$ be a fct of 2 variables & differentiable.
dx AND dy are independent.
differential dz is:

x $$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

if $dx = \Delta x = x-a$ & $dy = \Delta y = y-b$, we get:
$\underline{dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)}$
So
$$\underline{L(x,y) = f(a,b)+dz} \quad \text{so} \quad \underline{f(x,y) \approx f(a,b)} + dz$$
Δz is change in f when x changes by Δx & y changes by Δy
dz is change in linear approx when $dx = \Delta x$ and $dy = \Delta y$.

$dz \approx \Delta z$ near (a,b) so dz gives good estimates for errors in computed values obtained from measurements that have small errors

You can do this for fcts of >2 var.!
ex. if $w = f(x,y,z)$, then
$f(x,y,z) \approx L(x,y,z)$ where
$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$
is the linear approximation near (a,b,c)

x increment Δw corresponding to $\Delta x, \Delta y, \Delta z$ is
$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$

the differential is
$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$ where dx, dy, and dz are ind. variables.

ex.4: Find eqn to tangent plane to $z = xe^{xy}$ to $(2,0,2)$
$f_x(x,y) = xye^{xy} + e^{xy}$ $\quad f_y = x^2e^{xy}$ so $f_x(2,0) = 1$ and $f_y(2,0) = 4$.
$\therefore$ eqn of tangent plane is: $z-2 = 1(x-2) + 4(y-0)$ or
at $(2,0,2)$ $\quad z = x+4y$

ex.14: Find linear approximation for $\sqrt{y+\cos^2 x}$ at $(0,0)$
$f_x(x,y) = \frac{1}{2}(y+\cos^2 x)^{-1/2} \cdot 2\cos x \cdot -\sin x$ $\quad f_x(0,0) = 0$
$f_y(x,y) = \frac{1}{2}(y+\cos^2 x)^{-1/2}$ $\quad f_y(0,0) = 1/2$

$L(x) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$
$= 1 + \frac{1}{2}y$

ex. 33: Use differentials to estimate max. error in calculated area of the rectangle
$A = x \cdot y$
$dA = \frac{dA}{dx}dx + \frac{dA}{dy}dy = y\,dx + x\,dy$

Here, $x=30$ & $y=24$, $dx = 0.1$ and $dy = 0.1$, so max error is about $3.0 + 2.4 = 5.4$ cm.

ex. 34. Use differentials to estimate amount of metal
$V = \pi r^2 h$ $\quad \Delta V \approx dV = \frac{dV}{dr}dr + \frac{dV}{dh}dh$

$dV = 2\pi rh\,dr + \pi r^2 dh$. $r=2$, $h=10$, $dr = 0.05$, $dh = 0.2$
b/c bottom & top

ex. 46.
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq 0,0 \\ 0 & \text{if } (x,y) = 0,0 \end{cases}$$