09/23/15

14.1 Functions of Several Variables

Definition: A real-valued function of n variables, $x_1, x_2, \ldots, x_n$ is a function $f: D \to \mathbb{R}$ where

D is a subset of

$D \subset \mathbb{R}^n$. The domain (D) is a subset of

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \ \forall i = 1, \ldots, n\}$$

the set of ordered n-tuples of real numbers

- To each $(x_1, \ldots, x_n)$, f assigns a unique real number:

$$f(x_1, \ldots, x_n)$$

- If f is given by a formula and D is not specified, then D is the largest subset of $\mathbb{R}^n$ on which f is well defined.

- Let $z = f(x_1, \ldots, x_n)$. z is a dependent variable, while $x_1, \ldots, x_n$ are independent variables.

- The range of f is the set of all values f takes on.

Three points of view:

Since there is a one-to-one correspondence between points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ & their pos. vectors $\langle x_1, \ldots, x_n \rangle \in V^n$, we can think of $f: D \to \mathbb{R}$ in three ways:

- A function of n real variables $x_1, \ldots, x_n$
- A function of a single point variable $(x_1, \ldots, x_n) \in \mathbb{R}^n$
- A function of a single vector variable $\langle x_1, \ldots, x_n \rangle$

Linear Functions

$f: \mathbb{R}^n \to \mathbb{R}$ or $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ given by:

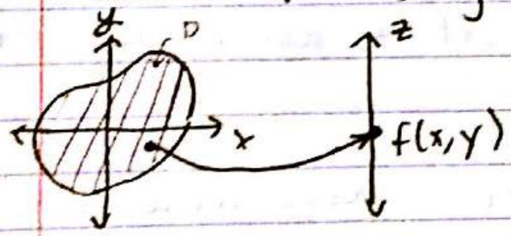
$$f(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n + b,$$

where $a_1, \ldots, a_n, b \in \mathbb{R}$ are constants, is a *linear* function of n variables.

- If $n=1$, $f(x) = ax+b$. Graph of f is the set of points (x,y) in the plane such that $y=f(x)$: a line
- If $n=2$, $f(x,y) = ax_1 + a_2 y + b$. The graph of f is the set of points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y) = a_1 x + a_2 y + b$, a plane.
- If $b=0$, let $\vec{a} = \langle a_1, \ldots, a_n \rangle$ and $\vec{x} = \langle x_1, \ldots, x_n \rangle$. Then we have a shorthand notation for:
  $f(x_1, \ldots x_n) = a_1 x_1 + \ldots + a_n x_n$:
  $f(\vec{x}) = \vec{a} \cdot \vec{x}$.

Functions of Two Variables

- Use an arrow diagram:



$z = f(x,y)$

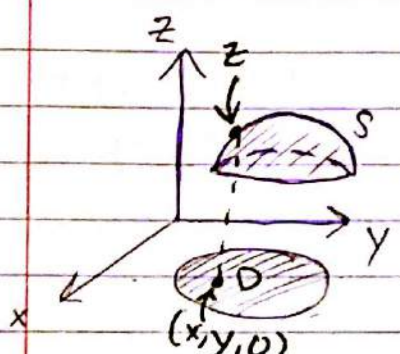
- Partially represented by table:

x \ y	.	.	.	b
.				
.				
a				$\leftarrow f(a,b)$
.				

$D = \text{dom} f$ is a subset of the plane $\mathbb{R}^2$, and we can sketch it.

Graphs: The graph of a function f of two variables is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x,y)$ and $(x,y) \in D$ = domain of f.

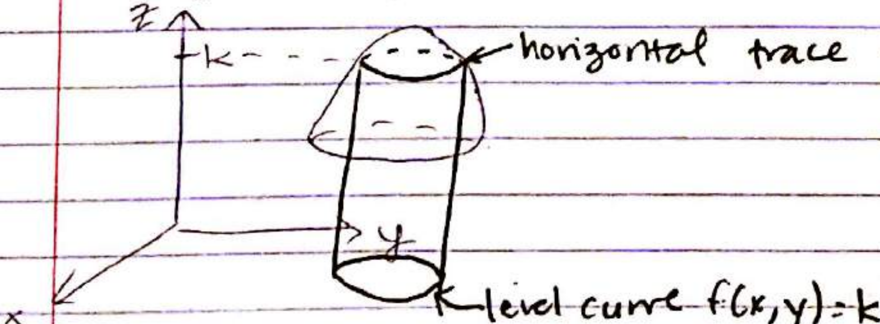
- a graph of a function with two variables is a surface S in $\mathbb{R}^3$ with equation $z = f(x,y)$
- we visualize this as lying above/below its doman D in the xy-plane.



(x, y, z)
$z = f(x,y)$

Level Curves: Level curves of a function of two variables are the curves (in $\mathbb{R}^2$) with equations $f(x,y) = k$ where $k \in$ (range of f) is constant.

- The level curve of $f(x,y) = k$ shows where the graph of f has height K
- A level curve $f(x,y) = k$ is the subset of D = dom $f \subset \mathbb{R}^2$ where f takes on the value k
- The level curve of $f(x,y) = k$ is the projection onto the xy-plane of the horizontal trace of the graph of corresponding to $z = k$



Functions of Three Variables:

- Domain is subset of $\mathbb{R}^3$
- Graph lies in $\mathbb{R}^4 \leftarrow \begin{cases} (x,y,z,w) \in \mathbb{R}^4 \text{ so that} \\ w = f(x,y,z) \ \& \ (x,y,z) \in \text{Dom} F \end{cases}$
- Its level <u>surfaces</u> are the surfaces (in $\mathbb{R}^3$) with the equations $f(x,y,z) = k$, where k is constant
  - f is constant on level surface

<u>Remark</u>:

- Function of n variables:
  Domain $\subset \mathbb{R}^n$ or V_n
  Range $\subset \mathbb{R}$

- Vector functions:
  Domain $\subset \mathbb{R}$
  Range $\subset V_n$

- In Ch. 16, we will consider vector fields, which are functions $f: \mathbb{R}^n \to V_n$
  ↳ this can be thought of as $f: V_n \to V_n$ in the same way $f(x_1, \dots, x_n) \to V_n$.

- In linear algebra, one considers linear transformations, which are linear functions $f: \mathbb{R}^m \to \mathbb{R}^n$ or $f: V \to W$ where V, W are vector spaces of any dimensions (even infinite.)