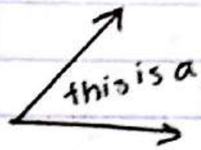
09/11/15

12.3 The Dot Product Lecture

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + \dots a_nb_n$$



where $\vec{a} = \langle a_1, a_2, a_3 \dots, a_n \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \dots, b_n \rangle$

☆ the dot product is a scalar value.

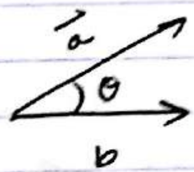
Properties:

① $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ ② $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

③ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

④ $(\lambda \vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda \vec{b})$

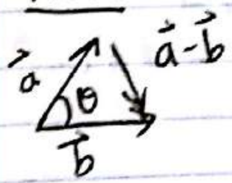
⑤ $0 \cdot \vec{a} = 0$



The angle between $\vec{a}$ and $\vec{b}$, $\theta \in [0, \pi]$

Then, $\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta}$

Proof



Law of Cosines

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

☆ this still works when $\theta = \pi$ or 0

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})(\vec{a} - \vec{b})$$

prop. 2, 3, 4 $= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} - \vec{b} \cdot \vec{a}$

prop. 1, 2 $= |a|^2 - 2\vec{a} \cdot \vec{b} + |b|^2 \therefore -2|\vec{a}||\vec{b}|\cos\theta = 2\vec{a} \cdot \vec{b}$

$\therefore \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Therefore,

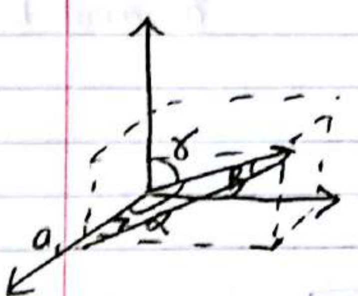
$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$$ (rearranging)

~~If~~ two vectors are $\perp$, if $\theta = \pi/2$

$0 \perp \vec{a}$ for all vectors $\vec{a}$

$$\therefore \boxed{\vec{a}\cdot\vec{b} = 0 \iff \vec{a} \perp \vec{b}}$$

Directional Angles & cosines



$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are directional cosins

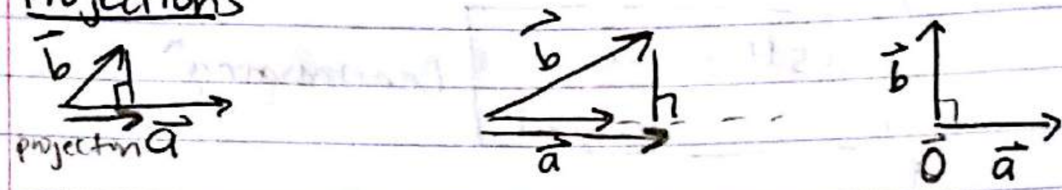
$$\cos\alpha = \frac{\vec{a}\cdot\hat{\imath}}{|\vec{a}||\hat{\imath}|} = \frac{a_1}{|\vec{a}|}$$

$$\cos\beta = \frac{a_2}{|\vec{a}|} \qquad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} = 1$$

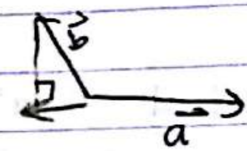
$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma \rangle$$

Projections



Projection of $\vec{b}$ onto $\vec{a}$:

$$\text{proj}_{\vec{a}}\vec{b}$$



Scalar projection of $\vec{b}$ onto $\vec{a}$ is a signed magnitude of the vector projection:

$$\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta$$, θ is $\angle$ between $\vec{a}$ and $\vec{b}$

positive if $0 \leq \theta < \pi/2$

0 if $\theta = \pi/2$

negative if $\pi/2 < \theta \leq \pi$

since $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

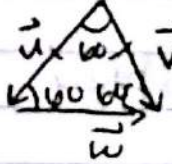
$$\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

also,

$$\text{proj}_{\vec{a}}\vec{b} = \underbrace{\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)}_{\text{comp}_{\vec{a}}\vec{b}}\underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector}} = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}}$$

example 11

tail to tail

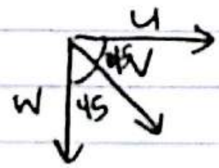


$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos 60 = |\vec{u}|^2 \cos 60$

$\vec{u}\cdot\vec{w} = |\vec{u}||\vec{w}|\cos 60 = |\vec{u}|^2 \cos 120$

$\vec{u}\cdot\vec{v} = \frac{1}{2}$ $\qquad \vec{u}\cdot\vec{w} = -\frac{1}{2}$

example 12



$\vec{u}\cdot\vec{v} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \boxed{\frac{1}{2}}$

$|\vec{u}| = 1$ $\qquad \vec{u}\cdot\vec{w} = 1\cdot 0 = \boxed{0}$

$|\vec{v}| = \frac{\sqrt{2}}{2}$

$|\vec{w}| = 1$

example 48

(a) when is $\text{comp}_a b = \text{comp}_b a$?

$\text{comp}_a b = |b|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ $\qquad$ when $|a| = |b|$

$\text{comp}_b a = |a|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ $\qquad \theta = 90°$

(b) when is $\text{proj}_a b = \text{proj}_b a$

$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$ $\qquad$ when $\vec{a} = \vec{b}$

or

$\theta = 90°$

example 54

If $\vec{r} = \langle x, y, z \rangle$, $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

show $(\vec{r}-\vec{a}) \cdot (\vec{r}-\vec{b}) = 0$ is a sphere

$$\vec{r}\cdot\vec{r} - \vec{r}\cdot\vec{b} - \vec{r}\cdot\vec{a} + \vec{a}\cdot\vec{b} = 0$$

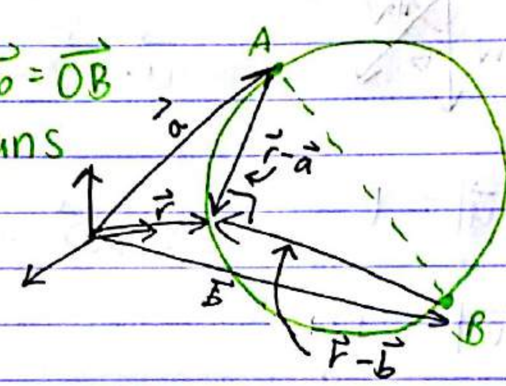
complete squares $(\vec{r}\cdot\vec{r} = |\vec{r}|^2)$

"COOL WAY"

Let $\vec{r} = \overrightarrow{OR}$, $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$

$(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b}) = 0$ means

$\vec{r}-\vec{a}$ is $\perp$ to $\vec{r}-\vec{b}$.



Therefore, R is on a sphere with diameter AB and radius $\frac{1}{2}|\vec{a}-\vec{b}| = \frac{1}{2}\sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$

example 61

Prove

$|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}|$

$= |\vec{a}||\vec{b}|\cos\theta$ ↑ BOOM DONE

$0 \leq \cos\theta \leq 1$