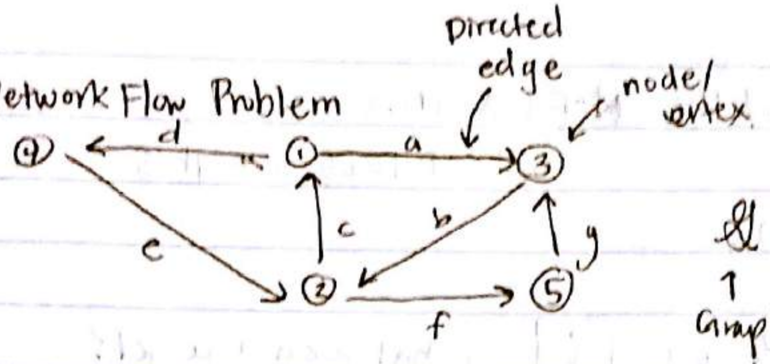
Network Flow Problem



$\mathcal{G} = \{V, \mathcal{E}\}$

Graph ↑ vertices ↑ edges ↑

Graph is set of vertices + edges.

$|V| = 5$ (number of nodes)

$|\mathcal{E}| = 7$ (number of ~~ver~~ edges)

$V = \{v_1, \dots, v_5\}$

$\mathcal{E} = \{(1,3), (3,2), (2,1), \dots\}$

What eqns can represent this graph? (assume no accumulation)

$$\left.\begin{array}{ll} c = a+d & \text{(node 1)} \\ d = e & \text{(node 4)} \\ b+e = c+f & \text{(node 2)} \\ a+g = b & \text{(node 3)} \\ f = g & \text{(node 5)} \end{array}\right\}$$ state has to satisfy these constraints.

Node - Edge Incidence Matrix

$$\begin{array}{c} \\ \text{Nodes} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{bmatrix} -1 & 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & +1 & -1 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix}_{5\times 7} = G$$

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$

⟵ edges ⟶

node edge is going into: +1

out of: −1

$$\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbb{1}^T G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

↑

sanity check yay

What you can tell w/ the N-EI Matrix:

- nodes of graph
- vertices of graph

· CAN'T TELL: weights.

If we multipled G by $\begin{bmatrix} a \\ b \\ \vdots \\ g \end{bmatrix}$, what would we get?

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix}$$

← in nullspace of B!

$$= \begin{bmatrix} -a+c-d \\ b-c+e-f \\ a-b+g \\ d-e \\ f-g \end{bmatrix}_{5\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \;!!$$

Being in nullspace
↳ conservation of packets

$G\vec{x} = \vec{0} \Rightarrow \vec{x} \in \mathcal{N}(G)$

You can solve for nullspace by row reducing smartly.

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

recover the system
↓

$$
\begin{bmatrix}
-1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$\Rightarrow 0g = 0 \Rightarrow$ <u>g is free</u>.

$-f + g = 0 \Rightarrow f = g$.

$-d + e - f + g = 0 \Rightarrow d = e$; <u>$e$ is free</u>.

$b - c + d - e = 0 \Rightarrow b = c$; <u>$c$ is free</u>.

$-a + c - d = 0 \Rightarrow a = -d + c$

$\dim(N(G)) = 3.$

$$
\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix}
=
\begin{bmatrix}
1 & -1 & 0 \\
1 & -1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} c \\ e \\ g \end{bmatrix}
$$

← forms basis for $N(G)$

$\uparrow \vec{w}_1 \quad \uparrow \vec{w}_2 \quad \uparrow \vec{w}_3$