Any Linear Transformation $\underset{\text{name}}{T}: \underset{\text{domain}}{\mathbb{R}^N} \to \underset{\text{range}}{\mathbb{R}^M}$ has a matrix representation.

T is linear if for every $\vec{x}, \vec{y} \in \mathbb{R}^N$ and every $\alpha, \beta \in \mathbb{R}$:

$T(\alpha\vec{x} + \beta\vec{y}) = T(\alpha\vec{x}) + T(\beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$.

(additivity prop) + (scaling prop) = superposition prop.

T has a matrix representation

$$\vec{e}_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ n \end{bmatrix}$$

$\underbrace{T(\vec{e}_1)}_{\in \mathbb{R}^m}$, so if $\vec{x} \in \mathbb{R}^N$, we should be able to find a matrix for T.

$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1) + \ldots + T(x_n\vec{e}_n) \\ &= x_1 T(\vec{e}_1) + \ldots + x_n T(\vec{e}_n) \\ &= \underbrace{\begin{bmatrix} T(\vec{e}_1) \ldots T(\vec{e}_n) \end{bmatrix}}_{\substack{(m\times n)\text{ matrix} \\ A}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} \end{aligned}$$

Four Fundamental Spaces of a Matrix A:

Column space of A: $R(A)$ [R for range]
$C(A)$ [C for column]

$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_N]$ $\qquad Ax = [\vec{a}_1 \ldots \vec{a}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{l=1}^{n} x_l a_l$

$C(A) = \{A\vec{x} \mid x \in \mathbb{R}^N\}$

$A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is in column space of A.

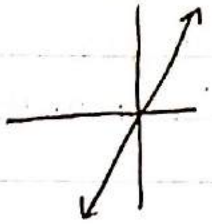
$\vec{b}$ = a linear combo of the columns of A.

If $A_{M\times N}$ and if $\vec{y} \in C(A)$, $\vec{y} \in \mathbb{R}^M$

so $C(A) \subseteq \mathbb{R}^m$.

When is $C(A)$ actually $\mathbb{R}^m$? w/equality if A has M linearly independent columns.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad C(A) = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \underbrace{\alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{C(B) \Rightarrow \mathbb{R}^2}$$

because columns of B are linearly independent
$B\vec{x} = \vec{d}$ has unique solution

Null space of A: $\mathcal{N}(A) = \{\vec{x} \in \mathbb{R}^N \mid A\vec{x} = \vec{0}\}$.

$$\mathcal{N}(A) \subseteq \mathbb{R}^N$$

$A\vec{x} = \vec{0}$
$A \in \mathbb{R}^{M \times N}$

$$\begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_m^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{\alpha}_1^T \cdot \vec{x} \\ \vec{\alpha}_2^T \cdot \vec{x} \\ \vdots \\ \vec{\alpha}_M^T \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

dot product!

This means $\vec{x} \perp \vec{\alpha}_\ell^T \quad \ell = 1, 2, \ldots, M$
If $\vec{x} = \vec{0}$ is the only solution to $A\vec{x} = \vec{0}$, A is full column rank M. (i.e. A has M linearly independent rows

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \mathcal{N}(A) = \left\{ \alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\} \quad \text{ie.} -3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 0.$$

(Solve $A\vec{x} = 0$)
get $[1 \;\; 3 \; \vdots \; 0]$

also, $\vec{x} = \vec{0}$ is in nullspace of every matrix A.

$$x \in \mathcal{N}(A) \text{ and } y \in \mathcal{N}(A) \Rightarrow \vec{z} = \vec{x} + \vec{y} \Rightarrow A\vec{z} = \overset{\vec{0}}{\cancel{A\vec{x}}} + \overset{\vec{0}}{\cancel{A\vec{y}}} = \vec{0}.$$

$N(A)$ is closed under linear combos of its elements.

Left Null space of A: $\{\vec{y} \mid \vec{y}^T A = \vec{0}\}$

$$\vec{y}^T A = [y_1 \ \dots \ y_m] \begin{bmatrix} \vec{\alpha}_1^T \\ \vdots \\ \vec{\alpha}_N^T \end{bmatrix} = \sum_{m=1}^{M} y_m \vec{\alpha}_m^T \rightarrow \text{linear combo of rows}$$

$\vec{y}^T A = \vec{0}^T \rightarrow A^T \vec{y} = \vec{0} \rightarrow N(A^T)$

$\vec{y}^T A = \vec{y}^T [\vec{a}_1 \ \dots \ \vec{a}_m] = [\vec{y}^T \vec{a}_1 \ \dots \ \vec{y}^T a_m] = [0 \ \dots \ 0]$

$\vec{y}$ is $\perp$ every of A

If $\vec{y} = \vec{0}$ is only solution to $\vec{y}^T A = \vec{0}^T$, then A has M linearly independent rows (A has full row rank)

Rank (A): # of linearly independent columns or # of linearly independent rows — equal

Row Space: $\{\vec{y}^T A \mid \vec{y} \in \mathbb{R}^M\} = C(A^T)$

set of all linear combos of rows of A.

Vector Spaces: A set V is a vector space if it satisfies all of the following properties:

- $x, y \in V \rightarrow x + y = y + x \in V$
- $x + (y + z) = (x + y) + z$
- $\exists \ \mathbb{O}$ s.t. $\vec{x} + \mathbb{O} = \vec{x}$
- $\alpha \vec{x} \in V, \ \alpha \in \mathbb{R}$
- $\alpha(\beta \vec{x}) = (\alpha\beta)\vec{x} \in V$
- $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$
- $\exists!$ negative s.t. $\vec{x} + (-\vec{x}) = \mathbb{O}$. — existence of negative
- $\mathbb{1}\vec{x} = \vec{x}$ — identity

Ex Space of all 2x2 upper triangular matrices is a vector space (but its components are matrices)

Basis: A set of vectors s.t.
(1) linearly independent
(2) they span the same

* span the space: all linear combinations together fill the space.

Dimension: # vectors in basis