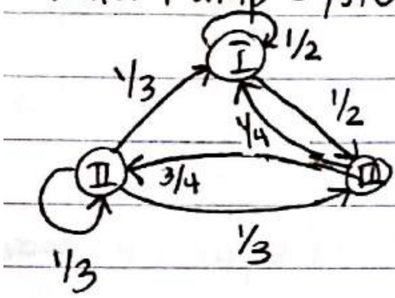
02/04

water Pump System:



State: $\vec{s}[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix}$ ← amt of water in I

statevector @ time n

$s[n]: \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $s[n+1]: \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$ $\vec{s}[n+1] = A\, s[n]$

$\vec{e}_1$ → unit vector

A ← state transition matrix

$$A: \begin{array}{c} \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 0 & 1/3 & 3/4 \\ 1/2 & 1/3 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{probability of moving to I} \\ \leftarrow \text{" II} \\ \leftarrow \text{" III} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{from I} & \text{from II} & \text{from III} \end{array} \end{array}$$

$$[\vec{a}_1, \vec{a}_2, \vec{a}_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [\vec{a}_1]$$

$$\vec{s}[n] = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \vec{s}[n+1] = \begin{bmatrix} \frac{1}{2}\alpha + \frac{1}{3}\beta + \frac{1}{4}\gamma \\ 0\alpha + \frac{1}{3}\beta + \frac{3}{4}\gamma \\ \frac{1}{2}\alpha + \frac{1}{3}\beta + 0\gamma \end{bmatrix} = A\, s[n]$$

(state evolution equation)

Q: If given $\vec{s}[n+1]$, can you determine $s[n]$?

A: Depends on A.

<u>Dfn</u>: A is said to be invertable if there exists a matrix, denoted by A^{-1} s.t. $A \cdot A^{-1} = I$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (of any dimension)

think of I as 1. $IA = A$. (in world of square matrices)

Suppose A^{-1} exists.

$A^{-1}s[n+1] = A^{-1}A\, s[n]$

$\rightarrow A^{-1}s[n+1] = s[n]$.

Matrix Inverses: * SQUARE MATRICES!

- $A^{-1}A = AA^{-1} = I$.
- If A has an inverse, it is unique

  Assume B & C are inverses of A:

  $BA = I \qquad AC = I \longrightarrow B(AC) = BI = B$

  $(BA)C = B \rightarrow C = B$.
- If A is invertible, $A\vec{x} = \vec{b}$ has a unique solution

  $A^{-1}A\vec{x} = A^{-1}\vec{b}$

  $\vec{x} = A^{-1}\vec{b}$ ✓
- If A has linearly dependent columns, it cannot have an inverse

  $A = [\vec{a}_1, \ldots, \vec{a}_N] \;\exists\; \vec{\alpha} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \in \mathbb{R}^N$ s.t. $A\vec{\alpha} = \vec{0}$, $\vec{\alpha} \neq \vec{0}$

- Assume an inverse exists:

  $A\vec{\alpha} = \vec{0}$

  $A^{-1}A\vec{\alpha} = A^{-1}\cdot\vec{0} \rightarrow \vec{\alpha} = \vec{0}$ contradicts linear dependence

3 Reservoir Example:

$$A = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 0 & 1/3 & 3/4 \\ 1/2 & 1/3 & 0 \end{bmatrix} \qquad A^{-1} = [\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3]$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$AA^{-1} = I = A[\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3] = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]$$

$$[A\vec{x}_1 \ A\vec{x}_2 \ A\vec{x}_3] = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]$$

$\hookrightarrow A\vec{x}_1 = \vec{e}_1$

$A\vec{x}_2 = \vec{e}_2$

$A\vec{x}_3 = \vec{e}_3$

$[A \mid I]$ (want to get A to look like I)

$$\left[\begin{array}{ccc|ccc} 1/2 & 1/3 & 1/4 & 1 & 0 & 0 \\ 0 & 1/3 & 3/4 & 0 & 1 & 0 \\ 1/2 & 1/3 & 0 & 0 & 0 & 1 \end{array}\right]$$

This method is called Gauss-Jordan elimination

$$= \left[\begin{array}{ccc|ccc} 1/2 & 1/3 & 1/4 & 1 & 0 & 0 \\ 0 & 1/3 & 3/4 & 0 & 1 & 0 \\ 0 & 0 & -1/4 & -1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1/2 & 0 & 0 & 3 & -1 & -2 \\ 0 & 1/3 & 0 & -3 & 1 & 3 \\ 0 & 0 & -1/4 & -1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -2 & -4 \\ 0 & 1 & 0 & -9 & 3 & 9 \\ 0 & 0 & 1 & 4 & 0 & -4 \end{array}\right]$$

YAY!

$$A^{-1} = \begin{bmatrix} 6 & -2 & -4 \\ -9 & 3 & 9 \\ 4 & 0 & -4 \end{bmatrix}$$

2x2 case

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\Delta = ad - bc$ determ. of A

Rank of a matrix $A \in \mathbb{R}^{N \times N}$

rank(A) = # linearly independent columns of A (pivots)

linearly independent rows of A for square matrices

A is rank deficient if it has fewer than N lin. indep cols (rows)

Range of A: span$\{\vec{a}_1, \ldots, \vec{a}_n\}$

If a is full rank, span$\{\vec{a}_1, \ldots, \vec{a}_n\} = \mathbb{R}^N$

$A\vec{x} = \vec{b}$